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Ordering trees with n vertices and matching number q by their largest Laplacian eigenvalues[☆]

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Abstract

Denote by $\mathscr{T}_{n,q}$ the set of trees with n vertices and matching number q. Guo [On the Laplacian spectral radius of a tree, Linear Algebra Appl. 368 (2003) 379–385] gave the tree in $\mathscr{T}_{n,q}$ with the greatest value of the largest Laplacian eigenvalue. In this paper, we give another proof of this result. Using our method, we can go further beyond Guo by giving the tree in $\mathscr{T}_{n,q}$ with the second largest value of the largest Laplacian eigenvalue.

MSC: 05C50

Keywords: Tree; Laplacian eigenvalues; Spectral radius; Matching number

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For $v \in V$, $N(v)$ denotes the set of all neighbors of vertex v in G, and the degree of v, written by $d(v)$, is the cardinality of $N(v)$. The adjacency matrix of a graph G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i and v_j are adjacent; $a_{ij} = 0$ otherwise. Let $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Denote by $P(G, x)$ the characteristic polynomial of $L(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are non-negative real numbers. We denote the largest eigenvalue of $L(G)$ by $\mu(G)$ and call it the Laplacian spectral radius of G. Two distinct edges in a graph G are independent if they do not have a common end vertex. A set of pairwise independent edges of G is called a matching in G, while a matching of maximum cardinality is a maximum matching in G. The matching number q of G is the cardinality of a maximum matching of G.

The investigation on the Laplacian spectral radius of graphs is an important topic in the theory of graph spectra (see, e.g., [2–13] and the references therein). Recently, the problem concerning graphs with maximal or minimal Laplacian spectral radius of a given class of graphs has been studied by many authors. Denote by $\mathscr{T}_n$ the set of trees on n vertices. Zhang and Li [13] and Guo [4] gave the first four trees in $\mathscr{T}_n$, ordered according to their Laplacian radii. Yu et al. [12] determined the fifth to eighth trees in the above ordering. Gutman [6] proved that the star has the largest Laplacian

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spectral radius in $\mathscr{T}_n$. Petrović and Gutman [11] proved the path has the smallest Laplacian spectral radius in $\mathscr{T}_n$. Guo [5] determined the first four graphs with the largest Laplacian spectral radius and the graph with the smallest Laplacian spectral radius among all unicyclic graphs on n vertices. Let $T_m^n (2m \leqslant n+1)$ denote the tree obtained from a star $K_{1,n-m}$ by joining $m-1$ pendant vertices of $K_{1,n-m}$ to $m-1$ isolated vertices by $m-1$ edges. Guo [4] proved that among all trees with n vertices and matching number q, T_q^n has the largest Laplacian spectral radius. Hong and Zhang [7] determined the tree with the largest Laplacian spectral radius among all the trees with n vertices and k pendant vertices. As a corollary, Hong and Zhang [7] gave a simple proof of the above result of Guo on trees with n vertices and matching number q.

Denote by $\mathscr{T}_{n,q}$ the set of trees with n vertices and matching number q. In this paper, we give another proof of the above result of Guo on trees with n vertices and matching number q. Using our method, we can go further beyond Guo by giving the tree in $\mathscr{T}_{n,q}$ with the second largest Laplacian spectral radius.

2. Preliminaries

Denote by ${}_{(i,\alpha)}T_{(j,\beta)}$ the tree in $\mathscr{T}_{n,q}$ shown in Fig. 1, where i, j, α and β are all nonnegative integers, and we assume that $\alpha \leqslant \beta$ when $i = j$. In particular, ${}_{(0,0)}T_{(0,n-2)}$ is the star $K_{1,n-1}$, and ${}_{(0,0)}T_{(q-1,n-2q)}$ is T_q^n. The terminology not defined here can be found in [1,2].

Let $K(G) = D(G) + A(G)$, and let $v(G)$ denote the spectral radius of $K(G)$. For a connected graph G, $K(G)$ is non-negative and irreducible, and by the Perron–Frobenius theory of non-negative matrices, $v(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $v(G)$. We shall refer to such an eigenvector as the Perron vector of G. Very recently, Hong and Zhang gave the following useful result.

Lemma 1 (*Hong and Zhang [7]*). *Let G be a connected graph and $v(G)$ be the spectral radius of $D(G) + A(G)$. Let u, v be two vertices of G and $d(v)$ be the degree of vertex v. Suppose $v_1, v_2, \ldots, v_s \in N(v)\backslash N(u) (1 \leqslant s \leqslant d(v))$ and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of $D(G) + A(G)$, where x_i corresponds to the vertex $v_i (1 \leqslant i \leqslant n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges $uv_i (1 \leqslant i \leqslant s)$. If $x_u \geqslant x_v$, then $v(G) < v(G^*)$.*

Grone et al. [3] (also see [9]) showed that if G is a bipartite graph then $K(G)$ and $L(G)$ are unitarily similar. This implies the following lemma.

Lemma 2. *If G is a bipartite graph, then $K(G)$ and $L(G)$ have the same spectrum.*

Moreover, Pan [10] proved that $\mu(G) \leqslant v(G)$ with equality if and only if G is a bipartite graph. When G is a tree, it follows from Lemma 2 that the investigation on the Laplacian spectrum of G may be reduced to the investigation of spectrum of $K(G)$.

By Lemmas 1 and 2, we obtain easily the following two lemmas which may be regarded as immediate consequences of Lemmas 1 and 2.

Lemma 3. *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \cap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v, and adding a new pendant edge to $u(=v)$. Then $v(G) < v(G^*)$. In particular, if G is a tree, then $\mu(G) < \mu(G^*)$.*

Proof. We use x_u and x_v to denote the components of the Perron vector of G corresponding to u and v. Suppose that $N(u) = \{v, v_1, \ldots, v_s\}$ and $N(v) = \{u, u_1, \ldots, u_t\}$. Since $e = uv$ is a non-pendant edge of G, it follows that $s, t \geqslant 1$.

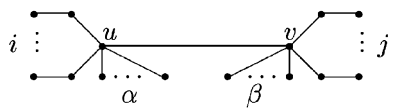

Fig. 1. ${}_{(i,\alpha)}T_{(j,\beta)}$.

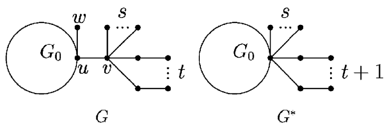

Fig. 2. G, G^*.

If $x_u \geqslant x_v$, let

$$G' = G - \{vu_1, \ldots, vu_t\} + \{uu_1, \ldots, uu_t\}.$$

If $x_u < x_v$, let

$$G'' = G - \{uv_1, \ldots, uv_s\} + \{vv_1, \ldots, vu_s\}.$$

Obviously, $G' = G'' = G^*$. By Lemma 1, we have $\nu(G) < \nu(G^*)$.

When G is a tree, by Lemma 2, we have $\mu(G) < \mu(G^*)$.

This completes the proof. $\square$

Lemma 4. *Let G, G^* be the graphs shown in Fig. 2, where G_0 is a connected graph with at least two vertices, $s \geqslant 2$, $t \geqslant 0$, or $s = 1$, $t \geqslant 1$. Then $\nu(G) < \nu(G^*)$. In particular, if G is a tree, then $\mu(G) < \mu(G^*)$.*

Proof. We use x_u and x_v to denote the components of the Perron vector of G corresponding to u and v. Suppose that $N(u) = \{v, w, u_1, \ldots, u_k\}$ and $N(v) = \{u, v_{11}, \ldots, v_{1s}, v_{21}, \ldots, v_{2t}\}$, where

$$d(v_{11}) = \cdots = d(v_{1s}) = 1, \quad d(v_{21}) = \cdots = d(v_{2t}) = 2.$$

Since G_0 is a connected graph with at least two vertices, it follows that $k \geqslant 1$.

If $x_u \geqslant x_v$, let

$$G' = G - \{vv_{12}, \ldots, vv_{1s}, vv_{21}, \ldots, vv_{2t}\} + \{uv_{12}, \ldots, uv_{1s}, uv_{21}, \ldots, uv_{2t}\}.$$

If $x_u < x_v$, let

$$G'' = G - \{uu_1, \ldots, uu_k\} + \{vu_1, \ldots, vu_k\}.$$

Obviously, $G' = G'' = G^*$. By Lemma 1, we have $\nu(G) < \nu(G^*)$.

When G is a tree, by Lemma 2, we have $\mu(G) < \mu(G^*)$.

This completes the proof. $\square$

We refer to the procedures of obtaining G^* from G, described in Lemmas 3 and 4, as Operations 1 and 2 on the graph G, respectively.

Lemma 5 (*Merris [9]*). *Let G have at least one edge, and $\triangle(G) = \max\{d(v) | v \in V(G)\}$. Then $\mu(G) \geqslant \triangle(G) + 1$. For G a connected graph on $n > 1$ vertices, the equality holds if and only if $\triangle(G) = n - 1$.*

Lemma 6 (*Li and Zhang [8]*). *Let G be a graph and $m(u)$ be the average of the degrees of the vertices of G adjacent to u. Then*

$$\mu(G) \leqslant \max\left\{ \frac{d(u)(d(u) + m(u)) + d(v)(d(v) + m(v))}{d(u) + d(v)} \,\middle|\, uv \in E(G) \right\}.$$

Lemma 7 (*Yu et al. [12]*). *Let G_1 and G_2 be two graphs. If $P(G_1, x) < P(G_2, x)$ for $x \geqslant \mu(G_1)$, then $\mu(G_1) > \mu(G_2)$.*

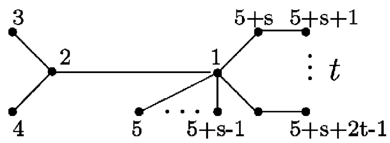

Fig. 3. ${}_{(0,2)}T_{(q-2,n-2q)}$.

Lemma 8. *Let $n \geqslant 2q+1$ and $q \geqslant 3$. Then*

$$\mu({}_{(1,1)}T_{(q-3,n-2q+1)}) \leqslant \mu({}_{(0,2)}T_{(q-2,n-2q)}),$$

and the equality holds if and only if $n=7$.

Proof. For $q=3$, ${}_{(0,2)}T_{(1,n-6)}$ is the graph $P_3^{1,n-6,2}$ of [12], and ${}_{(1,1)}T_{(0,n-5)}$ is the graph $P_3^{1,1,n-5}$ of [12] . When $n \geqslant 2q+2$, Lemma 8 follows from the proof of Lemma 2.10 of [12] (see p. 54). When $n=2q+1=7$, clearly ${}_{(0,2)}T_{(1,n-6)} = {}_{(1,1)}T_{(0,n-5)}$, and the equality of Lemma 8 holds.

Now we assume $q \geqslant 4$. Let $t=q-2$, $s=n-2q$. Then ${}_{(0,2)}T_{(q-2,n-2q)} = {}_{(0,2)}T_{(t,s)}$, and ${}_{(1,1)}T_{(q-3,n-2q+1)} = {}_{(1,1)}T_{(t-1,s+1)}$. For convenience, the vertices of ${}_{(0,2)}T_{(t,s)}$ are labeled anew by

$$1, 2, 3, 4, 5, \ldots, 5+s-1, 5+s, 5+s+1, \ldots, 5+s+2t-1,$$

shown as in Fig. 3, and the determinant of $xI - L({}_{(0,2)}T_{(t,s)})$ is denoted by $L_{s,t}$. Let $L_{s,t-i}$ $(0 \leqslant i \leqslant t)$ be the determinant obtained from $L_{s,t}$ by deleting the last $2i$ rows and the last $2i$ columns, and let

$$a(x) = \begin{vmatrix} x-2 & 1 \\ 1 & x-1 \end{vmatrix}, \quad b(x) = \begin{vmatrix} x-3 & 1 & 1 \\ 1 & x-1 & 0 \\ 1 & 0 & x-1 \end{vmatrix}.$$

Expanding the determinant $L_{s,t-i}$ $(0 \leqslant i \leqslant t-1)$ about the last two rows by Laplace Theorem for determinants, we have

$$L_{s,t-i} = a(x)L_{s,t-i-1} - (x-1)^{s+1}a(x)^{t-i-1}b(x). \tag{1}$$

Using (1) repeatedly, we have

$$\begin{aligned} L_{s,t} &= a(x)L_{s,t-1} - (x-1)^{s+1}a(x)^{t-1}b(x) \\ &= a(x)^2 L_{s,t-2} - 2(x-1)^{s+1}a(x)^{t-1}b(x) \\ &= \cdots\cdots \\ &= a(x)^t L_{s,0} - t(x-1)^{s+1}a(x)^{t-1}b(x). \end{aligned} \tag{2}$$

Using the properties of determinants, we have

$$L_{s,0} = \det(xI - L({}_{(0,2)}T_{(0,s)})) - t(x-1)^s b(x) = P({}_{(0,2)}T_{(0,s)}, x) - t(x-1)^s b(x). \tag{3}$$

Note that ${}_{(0,2)}T_{(0,s)}$ is the graph $T(2,s)$ of [4] (see p. 382). From [4] we have

$$P({}_{(0,2)}T_{(0,s)}, x) = x(x-1)^s(x^3 - (s+6)x^2 + (4s+9)x - s - 4). \tag{4}$$

Combining (2)–(4), we have

$$P({}_{(0,2)}T_{(q-2,n-2q)}, x) = L_{s,t} = x(x-1)^{n-2q}(x^2-3x+1)^{q-4}p(x),$$

where

$$\begin{aligned} p(x) = x^7 &- (n-q+10)x^6 + (10n-10q+36)x^5 - (36n-35q+56)x^4 + (56n-49q+34)x^3 \\ &- (36n-22q+4)x^2 + (10n-3q-1)x - n. \end{aligned}$$

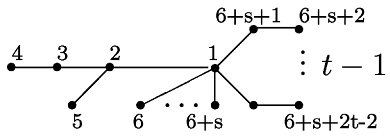

Fig. 4. ${}_{(1,1)}T_{(q-3,n-2q+1)}$.

Similarly, the vertices of ${}_{(1,1)}T_{(t-1,s+1)}$ are labeled anew by

$$1, 2, 3, 4, 5, 6, \ldots, 6+s, 6+s+1, 6+s+2, \ldots, 6+s+2t-2,$$

shown as in Fig. 4, and the determinant of $xI - L({}_{(1,1)}T_{(t-1,s+1)})$ is denoted by $L_{s+1,t-1}$. Let $L_{s+1,t-i}\,(1 \leqslant i \leqslant t)$ be the determinant obtained from $L_{s+1,t-1}$ by deleting the last $2(i-1)$ rows and the last $2(i-1)$ columns, and let

$$c(x) = \begin{vmatrix} x-3 & 1 & 0 & 1 \\ 1 & x-2 & 1 & 0 \\ 0 & 1 & x-1 & 0 \\ 1 & 0 & 0 & x-1 \end{vmatrix}.$$

Then

$$\begin{aligned} L_{s+1,t-1} &= a(x)L_{s+1,t-2} - (x-1)^{s+2}a(x)^{t-2}c(x) \\ &= a(x)^2 L_{s+1,t-3} - 2(x-1)^{s+2}a(x)^{t-2}c(x) \\ &= \cdots\cdots \\ &= a(x)^{t-1}L_{s+1,0} - (t-1)(x-1)^{s+2}a(x)^{t-2}c(x). \end{aligned} \tag{5}$$

Using the properties of determinants, we have

$$\begin{aligned} L_{s+1,0} &= \det(xI - L({}_{(1,1)}T_{(0,s+1)})) - (t-1)(x-1)^{s+1}c(x) \\ &= P({}_{(1,1)}T_{(0,s+1)}, x) - (t-1)(x-1)^{s+1}c(x). \end{aligned} \tag{6}$$

Note that ${}_{(1,1)}T_{(0,s+1)}$ is the graph $P_3^{1,1,s+1}$ of [12] (see p. 52). By (S4) of [12], we have

$$P({}_{(1,1)}T_{(0,s+1)}, x) = x(x-1)^s[x^5 - (s+10)x^4 + (7s+35)x^3 - (14s+52)x^2 + (8s+31)x - s - 6]. \tag{7}$$

Combining (5)–(7), we have

$$P({}_{(1,1)}T_{(q-3,n-2q+1)}, x) = L_{s+1,t-1} = x(x-1)^{n-2q}(x^2-3x+1)^{q-4}q(x),$$

where

$$\begin{aligned} q(x) = {} & x^7 - (n-q+10)x^6 + (10n-10q+36)x^5 - (36n-35q+56)x^4 + (57n-50q+30)x^3 \\ & - (39n-25q-8)x^2 + (11n-3q-8)x - n. \end{aligned}$$

Obviously, $\mu({}_{(0,2)}T_{(q-2,n-2q)})$ and $\mu({}_{(1,1)}T_{(q-3,n-2q+1)})$ are the maximum roots of the equations $p(x) = 0$ and $q(x) = 0$, respectively. Since

$$q(x) - p(x) = (n-q-4)x^3 - 3(n-q-4)x^2 + (n-7)x > 0$$

when $x > 3$, it follows that

$$P({}_{(1,1)}T_{(q-3,n-2q+1)}, x) > P({}_{(0,2)}T_{(q-2,n-2q)}, x)$$

when $x \geqslant \mu({}_{(0,2)}T_{(q-2,n-2q)}) > n-q$ (by Lemma 5). Hence by Lemma 7, we have

$$\mu({}_{(1,1)}T_{(q-3,n-2q+1)}) < \mu({}_{(0,2)}T_{(q-2,n-2q)}).$$

This completes the proof. □

3. Main results

When G is the tree ${}_{(i,\alpha)}T_{(j,\beta)}$, we use x_u and x_v to denote the components of the Perron vector, corresponding to the vertice u and v, respectively, of G.

Theorem 1 (*Guo [4]*). *Let $q \geqslant 2$, and T be a tree on n vertices with matching number q. Then $\mu(T) \leqslant r$, where r is the maximum root of the equation*

$$m(x) = x^3 - (n-q+4)x^2 + (3n-3q+4)x - n = 0.$$

The equality holds if and only if $T = T_q^n$.

Proof. Since $q \geqslant 2$, then $n-q \geqslant 2$. Let $X_l \subseteq \mathscr{T}_{n,q}$ be the set of all trees in which there exist exactly l pendant vertices. Then $\{X_l | n-2q+1 \leqslant l \leqslant n-q\}$ is a partition of $\mathscr{T}_{n,q}$. For any $T \in \mathscr{T}_{n,q}$, assume that M is a maximum matching of T. Then $|M| = q$ and there are three cases for a non-pendant edge $e = uv$ in T: (1) $e = uv$ is an M-saturated edge; (2) $e = uv$ has exactly one M-saturated vertex; (3) $e = uv$ is not an M-saturated edge but both u and v are M-saturated vertices. For a tree $T \in X_l$, if $l < n-q$, then T must have a non-pendant edge $e = uv$ of case (1) or case (2). By carrying Operation 1 once, we can transform T into a tree $T_1 \in X_{l+1}$ such that edge e is a pendant edge and $\mu(T) < \mu(T_1)$. It follows that the tree $T \in \mathscr{T}_{n,q}$ with the largest Laplacian spectral radius must be in X_{n-q}. In any such tree each non-pendant vertex is adjacent to a pendant vertex, otherwise there is a non-pendant edge of case (1) or case (2), in which case T cannot have the largest Laplacian spectral radius, in view of Operation 1. Clearly, $T_q^n \in X_{n-q}$, and for any $T \in X_{n-q}$ such that $T \neq T_q^n$, by carrying Operation 2 repeatedly, we can transform T into T_q^n. By Lemma 4, we have $\mu(T) < \mu(T_q^n)$. By a direct calculation (also see [4]), we have

$$P(T_q^n, x) = x(x-1)^{n-2q}(x^2-3x+1)^{q-2}[x^3 - (n-q+4)x^2 + (3n-3q+4)x - n].$$

This implies that $r = \mu(T_q^n)$ is the maximum root of the equation $m(x) = 0$, and completes the proof. □

Theorem 2. *Let $q \geqslant 3$, $T \in \mathscr{T}_{n,q}$ and $T \neq T_q^n$. If $n = 2q$, then*

$$\mu(T) \leqslant \mu({}_{(1,1)}T_{(q-3,1)})$$

and the equality holds if and only if $T = {}_{(1,1)}T_{(q-3,1)}$; if $n > 2q$, then

$$\mu(T) \leqslant \mu({}_{(0,2)}T_{(q-2,n-2q)})$$

and the equality holds if and only if $T = {}_{(0,2)}T_{(q-2,n-2q)}$.

Proof. For any $T \in \mathscr{T}_{n,q}$ such that $T \neq T_q^n$, from the proof of Theorem 1, it is easy to see that T can be transformed into T_q^n by carrying the Operations 1 and 2 repeatedly. Let A denote the set of all trees in $X_{n-q} \backslash \{T_q^n\}$ which can be transformed into T_q^n by carrying Operation 2 once, and let B denote the set of all trees in X_{n-q-1} which can be transformed into T_q^n by carrying Operation 1 once. It follows from Lemmas 3 and 4 that the trees with the second largest Laplacian spectral radius in $\mathscr{T}_{n,q}$ must be in $A \cup B$.

For any $T \in A$, from the definition of A, T must be ${}_{(i,\alpha)}T_{(j,\beta)}$ with $1 \leqslant i \leqslant j$, $i+j = q-2$, $\alpha \geqslant 1$, $\beta \geqslant 1$, $\alpha+\beta = n-2q+2$; or $i = 0$, $j = q-2$, $\alpha \geqslant 2$, $\beta \geqslant 1$, $\alpha+\beta = n-2q+2$. We consider the following two cases.

Case 1.1: $i \geqslant 1$. If $T \neq {}_{(1,1)}T_{(q-3,n-2q+1)}$, then either $x_u \geqslant x_v$ or $x_v > x_u$. In both the cases, we can apply Lemmas 1 and 2 to transform T to ${}_{(1,1)}T_{(q-3,n-2q+1)}$ such that

$$\mu(T) < \mu({}_{(1,1)}T_{(q-3,n-2q+1)}).$$

Hence, in this case, we have $\mu(T) \leqslant \mu({}_{(1,1)}T_{(q-3,n-2q+1)})$ with equality if and only if $T = {}_{(1,1)}T_{(q-3,n-2q+1)}$.

Case 1.2: $i=0$. In this case, we have $n>2q$. If $T\neq {}_{(0,2)}T_{(q-2,n-2q)}$, then $\alpha\geqslant 3$. By Lemmas 1 and 2, we have

$$\mu(T)<\begin{cases}\mu({}_{(0,2)}T_{(q-2,n-2q)}) & \text{if } x_v\geqslant x_u,\\ \mu({}_{(1,1)}T_{(q-3,n-2q+1)}) & \text{if } x_u>x_v.\end{cases}$$

By Lemma 8, we have $\mu(T)<\mu({}_{(0,2)}T_{(q-2,n-2q)})$.

For $T\in B$, from the definition of B, T must be ${}_{(i,0)}T_{(j,n-2q)}$ with $i\geqslant 1$, $j\geqslant 0$, $i+j=q-1$. We consider the following three cases.

Case 2.1: $n-2q\geqslant 2$, or $n-2q=1$, $j\geqslant 1$. By carrying Operation 1 once, T can be transformed into ${}_{(i-1,2)}T_{(j,n-2q)}\in A$. By Lemma 3 and Case 1.2, we have

$$\mu(T)<\mu({}_{(i-1,2)}T_{(j,n-2q)})\leqslant\mu({}_{(0,2)}T_{(q-2,n-2q)}).$$

Case 2.2: $n-2q=1$, $j=0$. Then $T={}_{(q-1,0)}T_{(0,1)}$. By a similar argument as that in the proof of Lemma 8, we have

$$\begin{aligned}P({}_{(q-1,0)}T_{(0,1)},x)={}&x(x^2-3x+1)^{q-3}[x^6-(q+9)x^5+(8q+30)x^4\\&-(23q+45)x^3+(28q+30)x^2-(13q+9)x+2q+1].\end{aligned}$$

From the proof of Lemma 8, we have

$$\begin{aligned}P({}_{(0,2)}T_{(q-2,1)},x)={}&x(x^2-3x+1)^{q-3}[x^6-(q+9)x^5+(8q+29)x^4\\&-(22q+42)x^3+(26q+27)x^2-(13q+7)x+2q+1].\end{aligned}$$

Therefore

$$P({}_{(q-1,0)}T_{(0,1)},x)-P({}_{(0,2)}T_{(q-2,1)},x)=x^2(x^2-3x+1)^{q-3}[x(x-2)(x-q-1)+x-2]>0$$

when $x\geqslant\mu({}_{(0,2)}T_{(q-2,1)},x))>q+1$. Hence by Lemma 7, we have

$$\mu({}_{(q-1,0)}T_{(0,1)})<\mu({}_{(0,2)}T_{(q-2,1)}).$$

Case 2.3: $n-2q=0$. Then $T={}_{(i,0)}T_{(j,0)}$ with $1\leqslant i\leqslant j$. By Lemmas 1 and 2, we have $\mu(T)\leqslant\mu({}_{(1,0)}T_{(q-2,0)})$ with equality if and only if $T={}_{(1,0)}T_{(q-2,0)}$. For $q\geqslant 4$, by a similar argument as that in the proof of Lemma 8, we have

$$\begin{aligned}P({}_{(1,0)}T_{(q-2,0)},x)={}&x(x^2-3x+1)^{q-4}[x^7-(q+10)x^6+(10q+37)x^5\\&-(38q+60)x^4+(68q+35)x^3-(57q-4)x^2+(19q-4)x-2q].\end{aligned}$$

From the proof of Lemma 8, we have

$$\begin{aligned}P({}_{(1,1)}T_{(q-3,1)},x)={}&x(x^2-3x+1)^{q-4}[x^7-(q+10)x^6+(10q+36)x^5\\&-(37q+56)x^4+(64q+30)x^3-(53q-8)x^2+(19q-8)x-2q].\end{aligned}$$

Therefore

$$P({}_{(1,0)}T_{(q-2,0)},x)-P({}_{(1,1)}T_{(q-3,1)},x)=x^2(x^2-3x+1)^{q-4}[x^2(x-4)(x-q)+5x^2-(4q+4)x+4]>0$$

when $x\geqslant\mu({}_{(1,1)}T_{(q-3,1)})>q$. Hence by Lemma 7, we have

$$\mu(T)\leqslant\mu({}_{(1,0)}T_{(q-2,0)})<\mu({}_{(1,1)}T_{(q-3,1)}).$$

If $q=3$, then $n=2q=6$. By a direct calculation (or from [11]), we have

$$\mu({}_{(1,0)}T_{(1,0)})<\mu({}_{(1,1)}T_{(0,1)}).$$

Combining the above arguments, the proof is completed. □

From the proofs of Theorem 2 and Lemma 8, we have the following.

Remark 1. When $q \geqslant 4$, $\mu({}_{(1,1)}T_{(q-3,1)})$ is the maximum root of the equation

$$x^7 - (q+10)x^6 + (10q+36)x^5 - (37q+56)x^4 + (64q+30)x^3 - (53q-8)x^2 + (19q-8)x - 2q = 0$$

and $\mu({}_{(0,2)}T_{(q-2,n-2q)})$ is the maximum root of the equation

$$x^7 - (n-q+10)x^6 + (10n-10q+36)x^5 - (36n-35q+56)x^4 + (56n-49q+34)x^3 - (36n-22q+4)x^2 + (10n-3q-1)x - n = 0.$$

When $q = 3$, $\mu({}_{(0,2)}T_{(1,n-6)})$ is the maximum root of the equation

$$x^5 - (n+4)x^4 + (7n-7)x^3 - (14n-32)x^2 + (7n-10)x - n = 0.$$

Remark 2. Using our method, we can also give the tree in $\mathscr{T}_{n,q}$ with the third largest Laplacian spectral radius.

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